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KUHN-TUKER CONDITIONS AND DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROGRAMS WITH *n*-SET FUNCTIONS

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Abstract
In mathematical programming of the n-set functionsis is considered a framework where the Kuhn-Tucker conditions are equality relations. For a multiobjective fractional program involving generalized (ρ,b)-vex n-set functions there is defined a multiobjective fractional pro-gram with equality constrains and weak, direct and converse duality theorems are established.

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1. INTRODUCTION

In 1979 Morris [9] developed the first optimization theory of the set functions. He defined the convexity and differentiability notions for the set functions and established optimal conditions and Lagrange duality for nonlinear programming problems with set functions.. The Corley's paper [2] is very important because he initiated the study of the functions of several set variables (*n*-set functions), gave the notions of partial derivative and of derivative of a *n*-set function and establishes some optimality conditions for mathematical programs involving *n*-set functions.

The research topic direction introduced by Corley was developed by Zalmai [12,13], Lin [4,5], Preda [10,11], I.M. Stancu-Minasian [11], Mititelu, Preda[7,8] etc.

Let Γ^n be the *n*-fold product of a σ-algebra Γ of subsets of a given set X and the vector functions $f=(f_1,..,f_p)':\Gamma^n\to\mathbf{R}^p$, $g=(g_1,\dots,g_p)':\Gamma^n\to\mathbf{R*}^p$, $h=(h_1,\dots,h_q)'\to\ \mathbf{R}^q$ and $k=(k_1,\dots,k_m)':\ :\Gamma^n\to\mathbf{R}^m$ ($'$ is the transposition sign). In this paper is presented a framework where the efficiency conditions of Kuhn-Tucker type are equality relations. This idea is illustrated with the following multiobjective program generated by n-set functions:

$$\text{(PE)}\begin{cases} \textit{Minimize} \ \left(f_1(S),\ldots, f_p(S)\right) \\ \textit{subject to} \quad h(S) \leqq 0, k(S)=0, S \in \Gamma^n. \end{cases}$$

We consider the following multiobjective fractional program with mixed constraints:

$$\text{(PFE)}\begin{cases} \textit{Minimize} \ \left(\dfrac{f_1(S)}{g_1(S)},\ldots,\dfrac{f_p(S)}{g_p(S)}\right) \\ \textit{subject to} \quad h(S) \leqq 0, k(S)=0, S \in \Gamma^n. \end{cases}$$

For the program (PFE) is developed a duality through weak, direct and converse duality theorems, where the main constraints of the dual program are equality relations. Generalized (ρ, b)-vexity hypotheses for the functions of the program are used. These results are extended also for a multiobjective fractional program with inequality and equality (mixed) constraints involving n-set functions of the same type. We denote by D_{PFE} the domain of (PFE).

We remind that for the vectors $u=(u_1,\ldots,u_n)'$ and $v=(v_1,\ldots,v_n)'$ the relations $u=v$, $u<v$, $u \leqq v$, $u \leq v$ etc, are defined as

$$u=v \Leftrightarrow u_i = v_i,\, i=\overline{1,n};\ u \leqq v \Leftrightarrow u_i \leqq v_i,\, i=\overline{1,n};$$
$$u<v \Leftrightarrow u_i < v_i,\, i=\overline{1,n};\ u \leq v \Leftrightarrow u \leqq v \text{ and } u \neq v.$$

We write $u'v$ for the inner product $\sum_{i=1}^{n} u_i v_i$ of u and v, where ‘ is the transposition sign.

2. DEFINITIONS AND PRELIMINARIES

Let (X,Γ,μ) be a finite positive atomless measure space with $L_1(X,\Gamma,\mu)$ separable. Consider the pseudometric space (Γ^n, d), where Γ^n is the n-fold product of the σ-algebra Γ and d is the pseudometric on Γ^n defined by

$$d(S,T)=\left[\sum_{k=1}^{n}(\mu(S_k \Delta T_k)\right]^{1/2}.$$

Here $S=(S_1,\ldots,S_n), T=(T_1,\ldots,T_n)$ and Δ denotes the symmetric difference.

The set $\Omega \in \Gamma$ can be identified with its indicator function $I_\Omega \in L_\infty(X,\Gamma,\mu) \subset L_1(X,\Gamma,\mu)$ and then the σ-algebra Γ is identified with the subset

$\{I_\Omega \mid \Omega \in \Gamma\} \subset L_\infty(X,\Gamma,\mu)$. For $\varphi \in L_1(X,\Gamma,\mu)$ and $\Omega \in \Gamma$ the integral $\int_\Omega \varphi d\mu$ will be denoted by $\langle \varphi, I_\Omega \rangle$.

Definition 2.1 [9]. A set function $\varphi : \Gamma \to \mathbf{R}$ is said to be *differentiable* at $T \in \Omega$ if there exists $D\varphi(T) \in L_1(X,\Gamma,\mu)$, called the *derivative* of φ at *T*, such that

$$\varphi(S) = \varphi(T) + \langle D\varphi(T), I_S - I_T \rangle + \psi(S,T)$$

for all $S \in \Gamma$, where $\psi : \Gamma \times \Gamma \to \mathbf{R}$ is $O(d(S,T))$, that is,

$$\lim_{d(S,T)\to 0} \psi(S,T)/d(S,T) = 0 .$$

Definition 2.2 [2]. A *n*-set function $F : \Gamma^n \to \mathbf{R}$ admits a *partial derivative* at $S^0 = (S_1^0,...,S_n^0)$ with respect to variable $S_k (1 \le k \le n)$ if the function $\varphi(S) = F(S_1^0,...,S_{k-1}^0, S_k, S_{k+1}^0,..., S_n^0)$ admits the derivative $D\varphi(S_k^0)$, and we define $D_k F(S^0) = = D\varphi(S_k^0)$. The derivative of *F* at S^0 is $DF(S^0) = (D_1F(S^0),..., D_n(S^0))$.

Definition 2.3 [2]. For a vector set function $f = (f_1,..., f_p)' : \Gamma^n \to \mathbf{R}^p$, the *partial derivative* with respect to the variable S_k at S^0 is $D_k f(S^o) = (D_k f_1(S^0),..., D_k f_p(S^0))$.

Definition 2.4 [2] A *n*-set function $F : \Gamma^n \to \mathbf{R}$ is differentiable at S^0 if there exists $DF(S^0)$ and $\psi : \Gamma^n \times \Gamma^n \to \mathbf{R}$ such that

$$F(S) = F(S^0) + \sum_{k=1}^{n} \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle + \psi(S,S^0),$$

where $\psi(S,S^0)$ is $O(d(S,S^o))$.

Definition 2.5 [11]. Let $F : \Gamma^n \to \mathbf{R}$ differentiable at S^0, $b : \Gamma^n \times \Gamma^n \to \mathbf{R}_+$ and $\rho \in \mathbf{R}$.

1) *F* is said to be (ρ,b)-*vex* [*strictly* (ρ,b)-*vex*] at S^0 if for all $S \in \Gamma^n$ [$S \ne S^0$] we have

$$b(S,S^0)\big(F(S) - F(S^0)\big) \geqq [>] \sum_{k=1}^{n} \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle + \rho b(S,S^0) d^2(S,S^0)$$

2) *F* is said to be *pseudo* (ρ,b)-*vex* [*strictly pseudo* (ρ,b)-*vex*] at S^0 if for all $S \in \Gamma^n$ [$S \ne S^0$] we have:

$$\sum_{k=1}^{n} \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle \geqq \rho d^2(S,S^0) \Rightarrow b(S,S^0)F(S) \geqq [>] b(S,S^0)F(S^0).$$

3) *F* is *quasi* (ρ,b)-*vex* [*strictly quasi* (ρ,b)-*vex*] at S^0 if for all $S \in \Gamma^n$ [$S \ne S^0$] we have:

$$F(S) \leqq F(S^0) \Rightarrow b(S,S^0) \sum_{k=1}^{n} \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle \leqq [<] - \rho b(S,S^0) d^2(S,S^0).$$

4) [7] F is said to be *monotonic quasi* (ρ, b)*-vex* at S^0 if for all $S \in \Gamma^n$ we have

$$F(S) = F(S^0) \Rightarrow b(S, S^0)\sum_{k=1}^{n}\langle D_k F(S^0), I_{S_k} - I_{S_k^0}\rangle = -\rho b(S, S^0) d^2(S, S^0).$$

Let us denote by D the domain of (PE) and let $P = \{1, ..., p\}$, $Q = \{1, ..., q\}$ and $M = \{1, ..., m\}$.

Definition 2.6 [3] A point $S^0 \in D$ is an *efficient solution* (Pareto minimum) for (PE) if there exists no $S \in D, S \neq S^0$, such that $f(S) \leq f(S^0)$.

For a multiobjective program with inequality constraints involving *n*-set functions Corley[2] defined the notion of the regular feasible solution and established necessary conditions for the efficiency of this solution. Mititelu and Preda adapted these two notions for the multiobjective program (PE) with mixed constraints as follows.

Definition 2.7 [8] A point $S^0 \in D$ is a *regular feasible solution* for (PE) if *h* and *k* are differentiable at S^0 and there exists $T \in \Gamma^n$ such that for all $j \in Q$ and $s \in M$ we have

$$(\mathrm{R})\begin{cases} h_j(S^0) + \sum_{k=1}^{n}\langle D_k h_j(S^0), I_{T_k} - I_{S_k^0}\rangle < 0, \\ \sum_{s=1}^{m}\langle D_k k_s(S^0), I_{T_k} - I_{S_k^0}\rangle \leqq 0. \end{cases}$$

Lemma 2.1 (Mititelu, Preda [8]). Let S^0 be a regular efficient solution of (PE) and let f, h and k be differentiable at S^0. Then there exist $u^0 \in \mathbf{R}^p, v^0 \in \mathbf{R}^q$ and $w^0 \in \mathbf{R}^m$ such that

$$\langle u^{0\prime} D_k f(S^0) + v^{0\prime} D_k h(S^0) + w^{0\prime} k(S^0), I_{S_k} - I_{S_k^0}\rangle \geqq 0 \qquad (2.1)$$

$$\forall S_k \in \Gamma, 1 \leqq k \leqq n$$

$$u^0 \geq 0, u^{0\prime} e = 1, e = (1, ..., 1) \in \mathbf{R}^p$$

$$v^0 \geqq 0, v^{0\prime} g(S) = 0, w^{0\prime} h(S) = 0.$$

In *n*-set programing these relations are considered efficiency conditions of Kuhn-Tucker type for the multiobjective program (PE) involving *n*-set functions.

3. KUHN-TUCKER EFFICIENCY CONDITIONS AS EQUALITY RELATIONS

In this section we establish efficiency conditions of Kuhn-Tucker type as equality relations for the program (PE) at a point $S^0 \in D$, using the regularity condition (R). The idea results from Theorem 2.2 by [6].

Theorem 3.1 (Necessary conditions KT for (PE)[11]). Let S^0 $(\neq \emptyset, \neq X)$ be a regular efficient (or weakly efficient) solution of the type (R). We also suppose that the functions

f, h and k are differentiable at S^0. Then there exist vectors $u^0 \in \mathbf{R}^p$, $v^0 \in \mathbf{R}^q$ and $w^0 \in \mathbf{R}^m$ such that the following efficiency conditions of Kuhn-Tucker type for (PE) at S^0 are satisfied :

$$\text{(KTE)}\begin{cases} u^{0}{}' D_k f(S^o) + v^{0}{}' D_k h(S^0) + w^{0}{}' D_k k(S^0) = 0, \ k = \overline{1,n} \\ v^{0}{}' h(S^0) = 0, \ v^0 \geqq 0 \\ u^0 \geq 0, \ e' u^0 = 1. \end{cases}$$

Proof. It is sufficient to analyse carefully the relation (2.1). We denote

$$C_k = u^{0}{}' D_k f(S^0) + v^{0}{}' D_k h(S^0) + w^{0}{}' D_k k(S^0)$$

and then, the relation (2.1), for each k, succesively becomes

$\langle C_k, I_{S_k} - I_{S_k^0} \rangle \geqq 0, \ \forall S_k \in \Gamma,$

$\langle C_k, I_{S_k} \rangle \geqq \langle C_k, I_{S_k^0} \rangle, \ \forall S_k \in \Gamma,$

$\int_{S_k} C_k d\mu \geqq \int_{S_k^0} C_k d\mu, \forall S_k \in \Gamma,$

$C_k \mu(S_k) \geqq C_k \mu(S_k^0), \forall S_k \in \Gamma,$

$$C_k[\mu(S_k) - \mu(S_k^0)] \geqq 0, \forall S_k \in \Gamma. \tag{3.1}$$

Particularly, for $S_k = \emptyset$ we have $\mu(S_k) = 0$ and from relation (3.1) it results $C_k \leqq 0, \ k = \overline{1,n}$. For $S_k = X$ and having $S_k^0 \subset X$, therefore $\mu(S_k^0) < \mu(X)$, then from (3.1) it results $C_k \geqq 0, k = \overline{1,n}$. Finally we obtain $C_k = 0, k = \overline{1,n}$ and so, the relation (2.1) is equivalent with the first relation of (KTE).

Zalmai established for the next multiobjective fractional program with inequality consraints:

$$\text{(PF)}\begin{cases} Minimize\left(\dfrac{f_1(S)}{g_1(S)}, \dots, \dfrac{f_p(S)}{g_p(S)}\right) \\ subject\ to\ \ h(S) \leqq 0, S \in \Gamma^n \end{cases}$$

((PFE) whitout the constraint $k(S) = 0$)) the following necessary efficiency conditions:

Lemma 3.1 (Zalmai[13, 2002]). Assume that $f_i, g_i, i \in P$ and $h_j \in Q$ are differentiable at $S^0 \in \Gamma^n$ and for each $i \in P$ there exist $\bar{S}^i \in \Gamma^n$ such that

$$h_j(S^0) + \sum_{k=1}^{n} \langle D_k h_j(S^0, I_{\bar{S}k} - I_{S_k^0} \rangle < 0$$

and for each $l \in P \setminus \{i\}$,

$$\sum_{k=1}^{n} \langle g_i(S^0) D_k f_l(S^0) - f_i(S^0) D_k g_l(S^0), I_{\bar{S}_k} - I_{S_k^0} \rangle < 0.$$

If S^0 is an efficient solution of (PF), then there exist $u^0 \in \mathbf{R}^p$ and $v^0 \in \mathbf{R}^q$ such that

$$\sum_{k=1}^{n} \langle \sum_{i=1}^{p} u_i^0 [g_i(S^0) D_k f_i(S^0) - f_i(S^0) D_k g_i(S^0)] + \sum_{j=1}^{q} v_j^0 Dh(S^0), I_{S_k} - I_{S_k^0} \rangle \geqq 0, \qquad (3.2)$$

$\forall S \in \Gamma^n$

$v_j^0 h_j(S^0) = 0, \; u^0 \geq 0, \; e'u^0 = 1, \; v^0 \geqq 0$.

Proposition 3.2 If $S^0 (\neq \emptyset, \neq X)$ is an efficient solution of (PF) then

$$c_k = \sum_{i=1}^{p} u_i^0 [g_i(S^0) D_k f(S^0) - f_i(S^0) D_k g_i(S^0)] + \sum_{j=1}^{q} v_j^0 D_k h_j(S^0) = 0.$$

Proof. Relation (3.2) becomes

$$\sum_{k=1}^{n} \langle c_k, I_{S_k} - I_{S_k^0} \rangle \geqq 0, \; \forall S_k \in \Gamma. \qquad (3.3)$$

Particularly, for $S_1 = S_1^0, ..., S_{k-1} = S_{k-1}^0 ; S_{k+1} = S_{k+1}^0, ..., S_n = S_n^0$ relation (3.3) becames

$$\langle c_k, I_{S_k} - I_{S_k^0} \rangle \geqq 0, \forall S_k \in \Gamma,$$

which implies $c_k = 0$. But the index k is arbitrarily choosen, Then $c_k = 0, \forall \, k = \overline{1, n}$.

Remark Relation (3.2) and $c_k = 0, k = \overline{1, n}$ are equivalent.

Consequently, according to Definition 2.7, Lemma 2.1, Proposition 3.2 and Remark, Definition 2.7 and Lemma 3.1, adapted for the program (PFE), have the following forms

Definition 2.8 A point $S^0 \in D_{PFE}$ is a regular feasible solution in Zalmai's sense for (PFE) if $f_i, g_i, i \in P$, $h_j, j \in Q$ and $k_s, s \in M$ are differentiable at $S^0 \in \Gamma^n$ and for each $i \in P$ there exist $\bar{S}^i \in \Gamma^n$ such that

$$h_j(S^0) + \sum_{k=1}^{n} \langle D_k h_j(S^0), I_{\bar{S}_k} - I_{S_k^0} \rangle < 0, \forall j \in Q,$$

$$\sum_{k=1}^{n} \langle D_k k_l(S^0), I_{\bar{S}_k} - I_{S_k^0} \rangle \leqq 0, \forall l \in M$$

and for each $l \in P \setminus \{i\}$,

$$\sum_{k=1}^{n} \langle g_i(S^0) D_k f_l(S^0) - f_i(S^0) D_k g_l(S^0), I_{\bar{S}_k} - I_{S_k^0} \rangle < 0.$$

Definition 2.9 A point $S^0 \in D_{PFE}$ is said to be a nonsingular solution for (PFE) if $S^0 \neq \emptyset$ and $\neq X$.

Theorem 3.3 (Necessary efficiency conditions for (PFE)). Assume that $f_i, g_i, i \in P$, $h_j,\ j \in Q$ and $k_s, s \in M$ are differentiable at $S^0 \in D_{PFE}$ **.** If S^0 is a nonsingular, regular efficient solution for (PFE) in Zalmai's sense, then there exist $u^0 \in \mathrm{R}^p, v^0 \in \mathrm{R}^q$ and $w^0 \in \mathrm{R}^m$ such that

$$\text{(KTFE)}\begin{cases} \sum_{i=1}^{p} u_i^0 [g_i(S^0) D_k f_i(S^0) - f_i(S^0) D_k g_i(S^0)] + \\ \qquad + \sum_{j=1}^{q} v_j^0 D_k h_j(S^0) + \sum_{s=1}^{m} w_s^0 D_k k_s(S^0) = 0 \\ 1 \leqq k \leqq n,\ v_j^0 h_j(S^0) = 0,\ \ j \in Q \\ u^0 \geqq 0, e' u^0 = 1,\ v^0 \geqq 0 . \end{cases}$$

The relations (KTFE) represent the efficiency conditions of Kuhn-Tucker type for (PFE) at S^0.

DUALITY BETWEEN (PFE) AND (DFE)

Let $\{Q_1,...,Q_r\}$ be a partition of Q, that is $Q_\alpha \subseteq Q, Q_\alpha \cap Q_\beta = \emptyset$ if $\alpha \neq \beta$, $\bigcup_{\alpha=1}^{r} Q_\alpha = Q$ and $\{M_1,...,M_r\}$ a similar partition of *M.*

We suppose that the functions $f_i, g_i, i \in P, h_j,\ j \in Q$ and $k_s, s \in M$ are differentiable on Γ^n. Then we associate to (PFE) the following dual multiobjective fractional program of maximum Pareto

$$\text{(DFE)}\begin{cases} \textit{Maximize} \quad \left(\dfrac{f_1(T)}{g_1(T)}, \dots, \dfrac{f_p(T)}{g_p(T)}\right) \\ \textit{subject to} \\ \sum_{i=1}^{p} u_i\, [g_i(T) D_k f_i(T) - f_i(T) D_k g_i(T)] + \sum_{j=1}^{q} v'_j D_k h_j(T) + \sum_{s=1}^{m} w_s D_k k_s = 0 \\ 1 \leqq k \leqq n, \quad v'_{Q_\alpha} h_{Q_\alpha}(T) + w'_{M_\alpha} k_{M'_\alpha} \geqq 0\,, \alpha = \overline{1,r} \\ T \in \Gamma^n, \quad u \geq 0, e'u = 1,\ v \geqq 0\,, \end{cases}$$

where

$$v'_{Q_\alpha} h_{Q_\alpha}(T) = \sum_{j \in Q_\alpha} v_j h_j(T), \quad w'_{M_\alpha} k_{M_\alpha}(T) = \sum_{s \in M_\alpha} w_s k_s(T).$$

We denote by $\pi(S)$ the value of the primal program (PFE), by $\delta(T, u, v, w)$ the value of the dual program (DFE) and be D_{DFE} the domain of (DFE). In what follows we develop a duality relation between the pair of multiobjective fractional programs (PFE) and (DFE) with weak, direct and converse duality theorems.

Theorem 4.1 (Weak duality). Let S and (T,u,v,w) be arbitrary feasible solutions of (PFE) and (DFE). Assume that:

a) For each $i \in P$, $f_i(T) > 0$, $g_i(T) > 0$, $g_i(S) > 0$;

b) For each $i \in P$, f_i is pseudo (ρ_i', b)-vex at T, $-g_i$ is pseudo (ρ_i'', b)-vex at T ;

c) For each $\alpha = \overline{1,r}$, $v_{Q_\alpha}' h_{Q_\alpha}$ is quasi (ρ_α''', b)-vex at T;

d) For each $\alpha = \overline{1,r}$, $w_{M_\alpha}' k_{M_\alpha}$ is monotonic quasi (ρ_α^4, b)-vex at T;

e) One of the functions $f_i, -g_i, \forall i \in P$ is strictly pseudo (ρ, b)-vex $(\rho = \rho_i',$ or $= \rho_i'')$, or one of $v_{Q_\alpha}' h_{Q_\alpha}$, $\forall \alpha$, is strictly quasi (ρ'', b)-vex at T;

f) $\sum_{i=1}^{p} u_i \left[\rho_i' g_i(T) + \rho_i'' f_i(T)\right] + \sum_{\alpha=1}^{r} (\rho_\alpha''' + \rho_\alpha^4) \geq 0.$

Then the relation $\pi(S) \leq \delta(T,u,v,w)$ is false.

Proof. From hypothesis b) it results:

$$\sum_{k=1}^{n} \langle D_k f_i(T), I_{S_k} - I_{T_k} \rangle \geqq -\rho_i' d^2(S,T) \Rightarrow b(S,T) f_i(S) \geqq b(S,T) f_i(T)\,, \qquad (4.1)$$

$$\sum_{k=1}^{n} \langle -D_k g_i(T), I_{S_k} - I_{T_k} \rangle \geqq -\rho_i'' d^2(S<T) \Rightarrow -b(S,T) g_i(S) \geqq -b(S,T) g_i(T)\,, \qquad (4.2)$$

For each $\alpha = \overline{1,r}$, according to c), we obtain

$$v_{Q\alpha}' h_{Q_\alpha}(S) \leqq v_{Q_\alpha}' h_{Q_\alpha}(t) \Rightarrow b(S,T) \sum_{k=1}^{n} \langle D_k v_{Q\alpha}' h_{Q\alpha}(T), I_{S_k} - I_{T_k} \rangle \leqq -\rho_\alpha''' b(S,T) d^2(S,T)$$

(4.3)

For each $\alpha = \overline{1,r}$, according to d), it results

$$w_{M_\alpha}' k_{M_\alpha}(S) = w_{M\alpha}' k_{M_\alpha}(T) \Rightarrow b(S,T) \sum_{k=1}^{n} \langle D_k w_{M_\alpha}' k_{M_\alpha}(T), I_{S_k} - I_{T_k} \rangle = -\rho_\alpha^4 b(S,T) d^2(S,T)$$

(4.4).

Taking into account the hypothesis e), for $S \neq T$, one of the second implications by (4.1), (4.2) or (4.3) is strictly: then means that $b(S,T) > 0$. In that follows, we consider these implications without the factor $b(S,T)$. Then, equivalently, we have

$$f_i(S) < f_i(T) \;\Rightarrow \sum_{i=1}^{p} \langle D_k f_i(T), I_{S_k} - I_{T_k} \rangle < -\rho_i' \, d^2(S,T)\,, \qquad (4.5)$$

$$-g_i(S) < -g_i(T) \;\Rightarrow\; \sum_{k=1}^{n} \langle -D_k g_i(T), I_{S_k} - I_{T_k} \rangle < -\rho_i'' d^2(S,T)\,. \qquad (4.6)$$

Now, we multiply (4.5) by $u_i g_i(T) \geqq 0$ (but $u'g(T) > 0$) and summ by $i \in P$, multiply (4.6) by $u_i f_i(T) \geqq 0$ $(u'f(T) > 0)$ and summ by $i \in P$ and then we summ, side by side, the two obtained inequalities.It results

$$\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - g_i(S) f_i(T)] < 0 \quad \Rightarrow$$

(4.7)

$$\Rightarrow \sum_{k=1}^{n} \langle \sum_{i=1}^{p} u_i [g_i(T) D_k f_i(T) - f_i(T) D_k g_i(T)], I_{S_k} - I_{T_k} \rangle < -\sum_{i=1}^{p} u_i [\rho_i' g_i(T) + \rho_i'' f_i(T)] d^2(S,T)$$

We summ now (4.3) by $\alpha = \overline{1,r}$ and summ also (4.5) by $\alpha = \overline{1,r}$ and then, we summ side by side, the two obtained inequalities. One obtain

$$\sum_{\alpha=1}^{r} \left[\left(v'_{Q_\alpha} h_{Q_\alpha}(S) + w'_{M_\alpha} k_{M_\alpha}(S) \right) - \left(v'_{Q_\alpha} h_{Q_\alpha}(T) + w'_{M_\alpha} k_{M_\alpha}(T) \right) \right] \leqq 0 \quad \Rightarrow \tag{4.8}$$

$$\Rightarrow \sum_{k=1}^{n} \langle \sum_{\alpha=1}^{r} \left(v'_{Q_\alpha} h_{Q_\alpha}(T) + w'_{M_\alpha} k_{M_\alpha}(T) \right), I_{S_k} - I_{T_k} \rangle \leqq -\sum_{\alpha=1}^{r} \left(\rho'''_\alpha + \rho^4_\alpha \right) d^2(S,T).$$

Now we summ, side by side, implications (4.7) and (4.8) and obtain:

$$\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - f_i(T) g_i(S)] + \sum_{j=1}^{q} v_j [h_j(S) - h_j(T)] + \sum_{s=1}^{m} w[k(S) - k(T)] < 0 \;\Rightarrow \; (4.8)$$

$$\Rightarrow \sum_{k=1}^{n} \langle \sum_{i=1}^{p} u_i [g_i(T) D_k f_i(S) - f_i(T) D_k g_i(T)] + \sum_{j=1}^{q} v_j D_k h_j(T) + \sum_{s=1}^{m} w_s D_k k_s , I_{S_k} - I_{T_k} \rangle <$$

$$< -\left(\sum_{i=1}^{p} u_i [\bar{\rho}_i g_i(T) + \hat{\rho}_i f(T)] + \sum_{\alpha=1}^{r} (\rho'''_\alpha + \rho^4_\alpha) \right) d^2(S,T).$$

.

Taking now into account the first constraint of the dual (DFE) and the hypothesis f), the second inequality by (4.9) becomes $0 < 0$, that is a false. Then, it results that the first inequality of (4.9) is false too and consequently, we have

$$\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - f_i(T) g_j(S)] + \sum_{j=1}^{q} v_j [h_j(S) - h_j(T)] + \sum_{s=1}^{m} w_s [k_s(S) - k_s(T)] \geqq 0, \;\; \forall u \geq 0$$

(4.10)

But taking into account the relations $S \in D_{PFE}$ and $(T,u,v,w) \in D_{DFE}$ we obtain:

$$\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - g_i(S) f_i(T)] \geqq 0, \; \forall u \geq 0, \; e'u = 1$$

and having $f_i(T)\, g_i(S) > 0, \forall i \in P$, we have

$$\sum_{i=1}^{p} u_i g_i(S) g_i(T) \left(\frac{f_i(S)}{g_i(S)} - \frac{f_i(T)}{g_i(T)} \right) \geqq 0, \;\; \forall u \geq 0, e'u = 1. \tag{4.11}$$

But $u_i g_i(S) g_i(T) \geqq 0$ (sometimes >0), $\forall i \in P$. Then, from (4.11), it results

$$\left(\frac{f_1(S)}{g_1(S)} - \frac{f_1(T)}{g_1(T)}, \dots, \frac{f_p(S)}{g_p(S)} - \frac{f_p(T)}{g_p(T)}\right) NO \le THAN \ \ (0, \dots, 0),$$

that is the inequality $\pi(S) \leqq \delta(T,u,v,w)$ is false.

Corollary 4.1 (Weak duality). Let S and (T,u,v,w) be arbitrary feasible solutions of (PFE) and (DFE), respectively. Assume satisfied the conditions a), b) of the Theorem 4.1 and the followings:

c') For each $\alpha = \overline{1,r}$, $\Gamma_\alpha = v'_{Q_\alpha} h_{Q_\alpha} + w'_{M_\alpha} k_{M_\alpha}$ is quasi (ρ'''_α, b)-vex at T;

d') one of the functions $f_i, -g_i, \forall i \in P$ is strictly pseudo (ρ, b)-vex $(\rho = \rho'_i, \rho''_i)$ at T, or one of $\Gamma_\alpha, \forall \alpha$, is strictly quasi (ρ'''_α, b)-vex at T;

f) $\sum_{i=1}^{p} u_i [\rho'_i g_i(T) + \rho''_i f_i(T)] + \sum_{\alpha=1}^{r} \rho'''_\alpha \geqq 0$.

Then the relation $\pi(S) \le \delta(T,u,v,w)$ is not true.

Theorem 4.2 (Direct duality). Let S^0 be a nonsingular regular efficient solution of (PFE) in Zalmai's sense and suppose satisfied the hypotheses of Theorem 4.1. Then there are vectors $u^0 \in \mathbf{R}^p$, $v^0 \in \mathbf{R}^q$ and $w^0 \in \mathbf{R}^m$ such that (S^0, u^0, v^0, w^0) is an efficient solution for the dual (DFE) and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

Proof. Because S^0 is a regular efficient solution of (PFE), according to Theorem 3.3 there are vectors $u^0 \in \mathbf{R}^p$, $v^0 \in \mathbf{R}^q$ and $w^0 \in \mathbf{R}^m$ such that the following relations are satisfied:

$$\sum_{i=1}^{p} u_i^{0'} [g_i(S^0) D_k f_i(S^0) - f_i(S^0) D_k g_i(S^0)] + \sum_{j=1}^{q} v_j^0 D_k h_j(S^0) + \sum_{s=1}^{m} w_s^0 D_k k_s = 0,$$

$$1 \leqq k \leqq n, \quad v_j^0 h_j(S^0) = 0, \ \forall j \in Q,$$

$$u^0 \ge 0, \ e' u^0 = 1, \ v^0 \ge 0.$$

Also $w_s^0 k_s(S^0) = 0, \forall s \in M$. Then from these relations it results that $(S^0, u^0, v^0, w^0) \in D_{DFE}$ and in addition,

$$\pi(S^0) = \left(\frac{f_1(S^0)}{g_1(S^0)}, \dots, \frac{f_{1i}(S^0)}{g_1(S^0)}\right) = \delta(S^0, u^0, v^0, w^0).$$

Because Theorem 4.2 contains the hypotheses of Theorem 4.1, the relation $\pi(S^0) \leqq \ \ \leqq \delta(S^0, u^0, v^0, w^0)$ is false. It follows that $(S^0, u,^0 v^0, w^0)$ is an efficient solution for (DFE).

Corollary 4.2 (Direct duality). Let S^0 be a nonsingular regular efficient solution of (PFE) and suppose satisfied the hypotheses of Corollary 4.1. Then there are vectors $u^0 \in \mathbf{R}^p, v^0 \in \mathbf{R}^q$ and $w^0 \in \mathbf{R}^m$ such that (S^0, u^0, v^0, w^0) is an efficient solution for the dual program (DPE) and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

Theorem 4.3 (Converse duality). Let (S^0, u^0, v^0, w^0) be an efficient solution of the dual program (DFE) and suppose that:

i) $\overline{S}$ is a nonsingular regular efficient solution of the primal program (PFE).

a^0) For each $i \in P$, $f_i(S^0) > 0$, $g_i(S^0) > 0$

b^0) For each $i \in P$, f_i is pseudo (ρ_i', b)-vex at S^0 and $-g_i$ is pseudo (ρ_i'', b)-vex at S^0.

c^0) For each $\alpha = \overline{1, r}$, $v'_{Q_\alpha} h_{Q_\alpha}$ is quasi (ρ_α''', b)-vex at S^0;

d^0) For each $\alpha = \overline{1, r}$, $w'_{M_\alpha} k_{M_\alpha}$ is monotonic quasi (ρ_α^4, b)-vex at S^0.

e^0) One of the functions $f_i, -g_i, \forall i \in P$ is strictly pseudo (ρ, b)-vex $(\rho = \rho_i', \rho_i'')$ at S^0 or one of $v'_{Q_\alpha} h_{Q_\alpha}, \forall \alpha$, is strictly quasi (ρ_α''', b)-vex at S^0;

f^0) $\sum_{i=1}^{p} u_i^0 \left[\rho_{ii}' g_i(S_0) + \rho_i'' f_i(S^0)\right] + \sum_{\alpha=1}^{r} (\rho_\alpha''' + \rho_\alpha^4) \geqq 0.$

Then $\overline{S} = S^0$ and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

Proof. Suppose, by absurdum, that $\overline{S} \neq S^0$ and we shall find a contradiction. Because $\overline{S}$ is a nonsingular regular efficient solution of (PFE) then, according to Theorem 3.3, there are vectors $\overline{u} \in \mathbf{R}^p, \overline{v} \in \mathbf{R}^q$ and $\overline{w} \in \mathbf{R}^m$ such that the next conditions of KTFE type are satisfied:

$$\sum_{i=1}^{p} \overline{u}_i [g_i(\overline{S}) D_k f_i(\overline{S}) - f_i(\overline{S}) D_k g_i(\overline{S})] + \sum_{j=1}^{q} \overline{v}_j D_k h_j(\overline{S}) + \sum_{s=1}^{m} \overline{w}_s D_k k_s(\overline{S}) = 0$$

$$1 \leqq k \leqq n, \quad \overline{u} \geq 0, \ e'\overline{u} = 1, \ \overline{v} \geqq 0, \ \overline{v}_j{}' h_j(\overline{S}) = 0, \ j \in Q.$$

Also we have $\overline{w}_s k_s(\overline{S}) = 0, \forall s \in M$. Then $(\overline{S}, \overline{u}, \overline{v}, \overline{w}) \in D_{DFE}$. The conditions a^0)--f^0) are particularly hypotheses of Theorem 4.1. Moreover, $\overline{S} \in D_{PFE}$ and $(S^0, u^0, v^0, w^0) \in D_{DFE}$. Following the proof of Theorem 4.1 we obtain that the relation $\pi(\overline{S}) \leq \delta(S^0, u^0, v^0, w^0)$ is false. Moreover, $\pi(\overline{S}) = \delta(\overline{S}, \overline{u}, \overline{v}, \overline{w})$. Therefore the relation $\delta(\overline{S}, \overline{u}, \overline{v}, \overline{w}) \leq \delta(S^0, u^0, v^0, w^0)$ is false. Then the maximal efficiency of (S^0, u^0, v^0, w^0) is contradicts. Therefore the supposition $\overline{S} \neq S^0$, above made, is false. It follows $\overline{S} = S^0$ and $\pi(S^0) = \delta(S^0, u^0, v^0)$.

Corollary 4.3 (Converse duality). Let (S^0, u^0, v^0, w^0) be an efficient solution of the dual program (DFE) and suppose satisfied the next conditions:

i) $\overline{S}$ is a nonsingular regular efficient solution of the primal program (PFE);

a^0), b^0) and f^0) of Theorem 4.3;

c′) For each $\alpha = \overline{1, r}$, $\Gamma_\alpha = v_{Q_\alpha}{}' h_{Q_\alpha} + w_{M_\alpha}{}' k_{M_\alpha}$ is quasi (ρ'''_α, b)-vex at S^0;

d') one of the functions $f_i, -g_i, \forall i \in P$ are strictly pseudo (ρ, b)-vex $(\rho = \rho'_i, \rho''_i)$ at S^0 or $\Gamma_\alpha, \forall\alpha$ is strictly quasi (ρ'''_α, b)-vex at S;

Then $\overline{S} = S^0$ and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

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